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# POTENTIAL-PLANE ANALOG SIMPLIFIES DESIGN OF NUCLEAR REACTOR CONTROL

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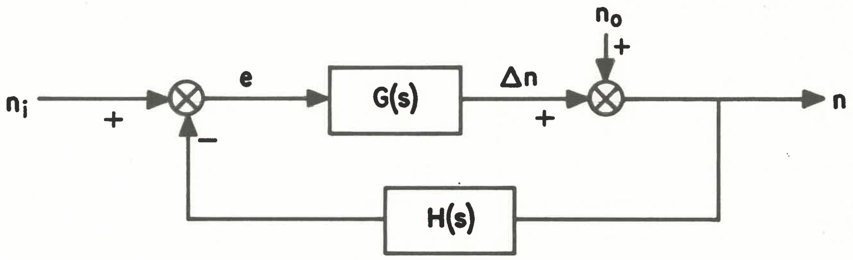

# POTENTIAL-PLANE ANALOG SIMPLIFIES DESIGN OF NUCLEAR REACTOR CONTROL

The potential-plane analog is based on the principle of using a voltage distribution on a uniform resistive sheet to represent a function of a complex variable. The equipotential lines may represent the log of the magnitude or the angle of a function of a complex variable. If two resistive sheets are used and the electrodes on one are orthogonal to the electrodes on the other, both the log magnitude and angle functions can be obtained simultaneously.

Although the principle of the potential-plane analog has been known for some time, it has seen little use in the past due to some severe limitations that made it difficult to construct a practical machine. An extension of potential-plane analog theory by M. L. Morgan has eliminated the previous restrictions with the result that a practical potential-plane analog is now available.

The theory and design of a practical potential-plane analog has been described previously (1,2,3) and the purpose of this paper is to show how a potential-plane analog can be used to simplify the analysis and synthesis of control systems, in particular, a nuclear reactor control system. In the following discussion, a nuclear reactor is described by a transfer function. This transfer function is then included as a component of the control system to be analysed with an ESIAC* potential-plane analog. Although simple transfer functions are used in this example to keep from drowning the analysis in a sea of mathematics, this should not be taken as a limitation on the ESIAC potential-plane analog, since it is capable of handling approximately twenty poles and zeros.

## REACTOR PHYSICS

A nuclear reactor is a system usually consisting of a moderator, fissionable fuel material, a means of removing heat, and a geometric structure suitable for maintaining a chain reaction. Figure 1 shows the basic components of one type of

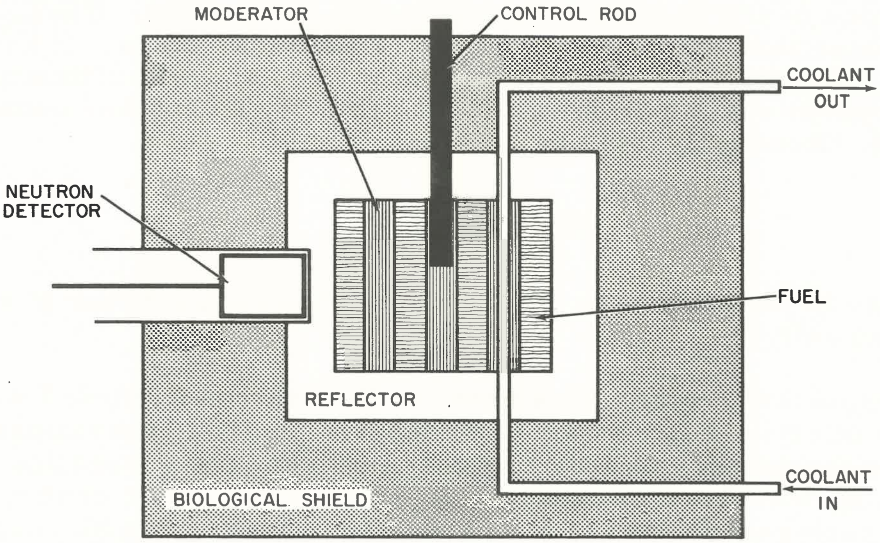

FIGURE 1. BASIC COMPONENTS OF A POWER REACTOR

power reactor. The fuel is located in the center section of the reactor and consists of uranium 235, in solid form. The moderator is intermingled with the fuel and is used to slow down the neutrons resulting from the fission process to where they possess the desired energy range. For this example, the moderator material is graphite and the neutrons are slowed down to the thermal energy range (0.026 ev).

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*ESIAC® Algebraic Computer, Manufactured by ESI Inc., Portland, Oregon.

The control rods are made of a neutron absorbing material such as cadmium, boron, or hafnium and they control the chain reaction by absorbing a percentage of neutrons produced in the fission reaction. In each fission reaction, the collision of one neutron with a $U^{235}$ nucleus will produce approximately 2.5 neutrons that are each capable of causing other fission reactions. However, a percentage of these neutrons are lost for various reasons and the control rods are used to absorb the excess so that the average number of neutrons available to produce another fission is approximately one.

The energy produced in the reactor core in the form of heat is transferred by the cooling system to an external plant which usually converts it to electrical energy. Outside of the reactor core is a reflector that helps to maintain the chain reaction by reflecting the neutrons back into the core that otherwise might be radiated and lost. A biographical shield surrounds the reactor and attenuates harmful radiation such as gamma rays and neutrons. The last basic component is a neutron detector which is used to monitor the level of neutron concentration in the core. A common type of neutron detector is an ionization chamber which gives an output current proportional to the number of fissions occuring per second. The power generated by the reactor is also proportional to the number of fissions per second. The relationship is:

$$3.10^{10} \text{ fissions/sec} = 1 \text{ watt} \tag{1}$$

Therefore, the neutron detector indicates the power level at which the reactor is operating.

For any specific reactor, an effective multiplication factor $k$ can be defined as the ratio of the number of neutrons in any one generation to the number of corresponding neutrons of the immediately receeding generation. Then, if $k$ is equal to, or slightly greater than unity, a chain reaction can take place. If $k$ is less than unity, a chain reaction cannot be self-regenerative, and will ultimately die out. In most reactor operations, $k$ will be very close to unity so another quantity, reactivity, is usually used. Reactivity $\Delta k$ is defined as:

$$\Delta k = \frac{k-1}{k} \approx k-1 \tag{2}$$

Thus, reactivity is the amount by which the multiplication factor of a specific reactor differs from unity.

Another important reactor parameter is the neutron lifetime $T$ which is defined as the average effective time between successive neutron generations in a specific reactor. Since the number of neutrons changes from one generation to the next by $\Delta k$, and assuming that there are initially $n$ neutrons per cubic centimeter, the rate of increase in each generation is $n\Delta k$. The rate of increase with respect to time is then:

$$\frac{dn}{dt} = \frac{\Delta k}{T} n \quad \text{neutrons/cm}^3\text{/sec} \tag{3}$$

Integrating this equation gives:

$$n = n_0 \epsilon^{\frac{(\Delta k)t}{T}} \quad \text{neutrons/cm}^3 \tag{4}$$

where $n_0$ is the initial concentration of neutrons, and $n$ is the concentration after time $t$. Equation (4) shows that the neutron concentration rises exponentially with time if $\Delta k$ is positive which is the case for $k$ greater than unity. Notice that the reactor period or time constant $\left(\frac{T}{\Delta k}\right)$ depends upon the effective lifetime $T$ and the reciprocal of the reactivity $\Delta k$.

Fortunately for the control system designer, the exponential response given in Equation (4) is modified by the fact that not all the neutrons created in the fission process are given off instantaneously. Approximately 0.75 percent of the neutrons are delayed, and their average lifetime is 10 seconds. Actually, these delayed neutrons are given off in six distinct groups at different times and in different quantities. The symbol $\beta$ is used to denote the total fraction of delayed neutrons (0.0075) and $\beta_i$ is the fraction of delayed neutrons in the $i$'th group. Similarly, the decay constant, which is the inverse of the average lifetime, is denoted by $\lambda_i$ with the average decay constant $\lambda$ equal to 0.1 sec$^{-1}$. The presence of the delayed neutrons imparts a desirable "inertia" effect to the exponential characteristic of the reactor. Consequently, it is practical to control a reactor automatically compared to the very difficult problem it would be without the delayed neutrons.

The instantaneous state of the reactor is described in terms of the value of the effective multiplication factor $k$. If $k = 1$, the reactor is said to be critical $k < 1$ subcritical, and $k > 1$ supercritical. If $k = 1.0075$, the reactor is said to be prompt critical. This means that the reactor can sustain a chain reaction without the use of the delayed neutrons. Generally, most control systems prevent $k$ from becoming greater than 1.0075 because the reactor characteristic becomes a rapid exponential when $k$ is greater than 1.0075.

## REACTOR TRANSFER FUNCTION

In the eyes of the control engineer, a reactor is a black box whose characteristics are specified by a transfer function. The reactor transfer function can then be incorporated with the transfer functions of the other system components to get the over-all system transfer function.

Since the behavior of the over-all system is specified by the system transfer function, the control engineer can then determine whether or not the system works properly, and if not, he can modify it so that it does.

It has been shown that a reactor can be described in the time domain by the set of differential equations shown below:

$$\frac{dn}{dt} = \frac{\Delta k - \beta}{T} n + \sum_{i=1}^{6} \lambda_i C_i \tag{5}$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{T} n - \lambda_i C_i \tag{6}$$

where $\beta_i$ is the number of delayed neutrons per cubic centimeter emitted in group i and the other symbols are the same as defined previously. For control purposes, the reactor is best described by a function in the frequency domain and this function $\frac{\Delta n}{\Delta k}$ may be found from the above equations with the aid of the Laplace Transform as shown in the appendix. The output quantity of the reactor transfer function is a change in neutron concentration $\Delta n$ and the input quantity is a change in reactivity $\Delta k$, which is due to a change in control rod position. The reactor transfer function is derived in the appendix and is:

$$\frac{\Delta n(s)}{\Delta k(s)} = \frac{n_0}{T} \frac{1}{s\left[1 + \sum_{i=1}^{6} \frac{\beta_i}{T(s+\lambda_i)}\right]} \tag{7}$$

and by expanding the denominator,

$$\frac{\Delta n(s)}{\Delta k(s)} = \frac{n_0}{T} \frac{\prod_{i=1}^{6}(s+\lambda_i)}{s\prod_{i=1}^{6}(s+r_i)} \tag{8}$$

where $s=\sigma+j\omega$ is the Laplace Transform variable and $\omega=2\pi f$. Using the values of $\lambda_i$ and $\beta_i$, given in reference (5), and a lifetime $T$ of $(T=10^{-4}_{\text{seconds}})$, the poles $r_i$ were found by factoring the sixth degree polynomial that results from expanding the denominator of Equation (7). The result is a reactor transfer function of:

$$\frac{\Delta n(s)}{\Delta k(s)} = \frac{n_0}{10^{-4}} \frac{(s+14)(s+1.61)(s+0.456)(s+0.151)(s+0.0315)(s+0.0124)}{(s+77)(s+13.4)(s+1.43)(s+0.34)(s+0.08)(s+0.015)} \tag{9}$$

where $n_0$ is the steady-state neutron concentration and $\Delta n$ is a change from the steady-state value.

Figures 2 and 3 show the Bode plots (magnitude and Angle versus log $f$) of the above transfer function as plotted with an ESIAC potential-plane analog. They are normalized with respect to $n_0$ with $(T=10^{-4}_{\text{seconds}})$.

Although the reactor transfer function shown in Equation (9) is not difficult to handle with an ESIAC potential plane analog, a simplified approximation suitable for most oontrol engineering is:

$$\frac{\Delta n}{\Delta k} = \frac{n_0}{10^{-4}} A \frac{(s+\bar{\lambda})}{s(s+\bar{r})} \tag{10}$$

The values of $A$, $\bar{\lambda}$ and $\bar{r}$ can be chosen to give Bode plots that resemble those of Equation (9) as accurately as possible.

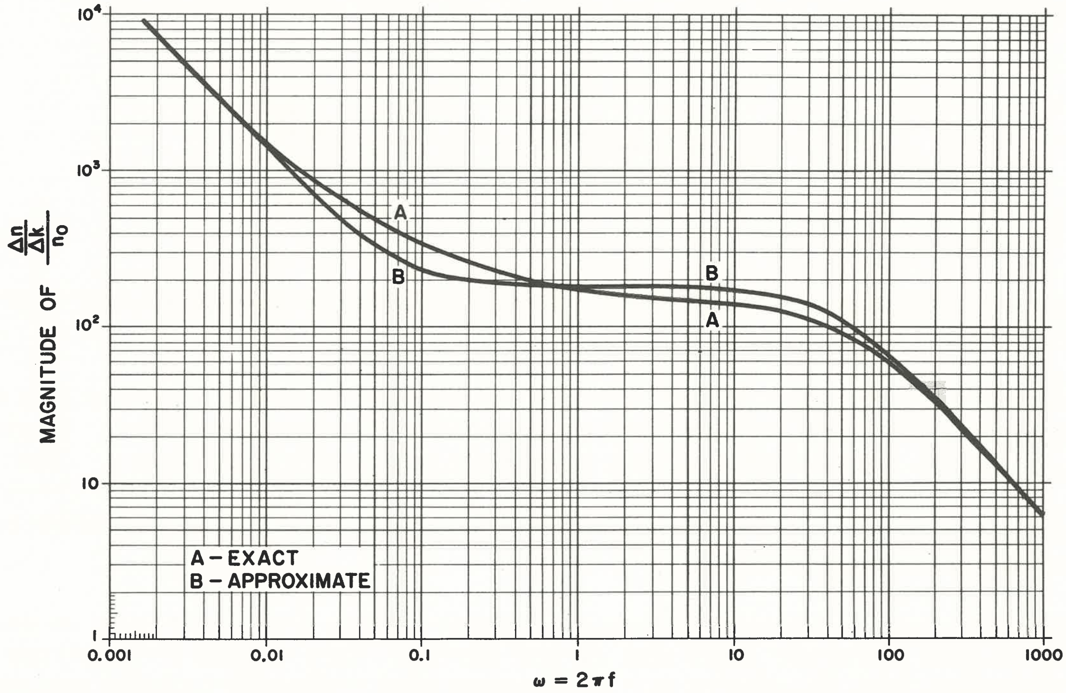

FIGURE 2. FREQUENCY RESPONSE OF ELEMENTARY REACTOR TRANSFER FUNCTION MAGNITUDE

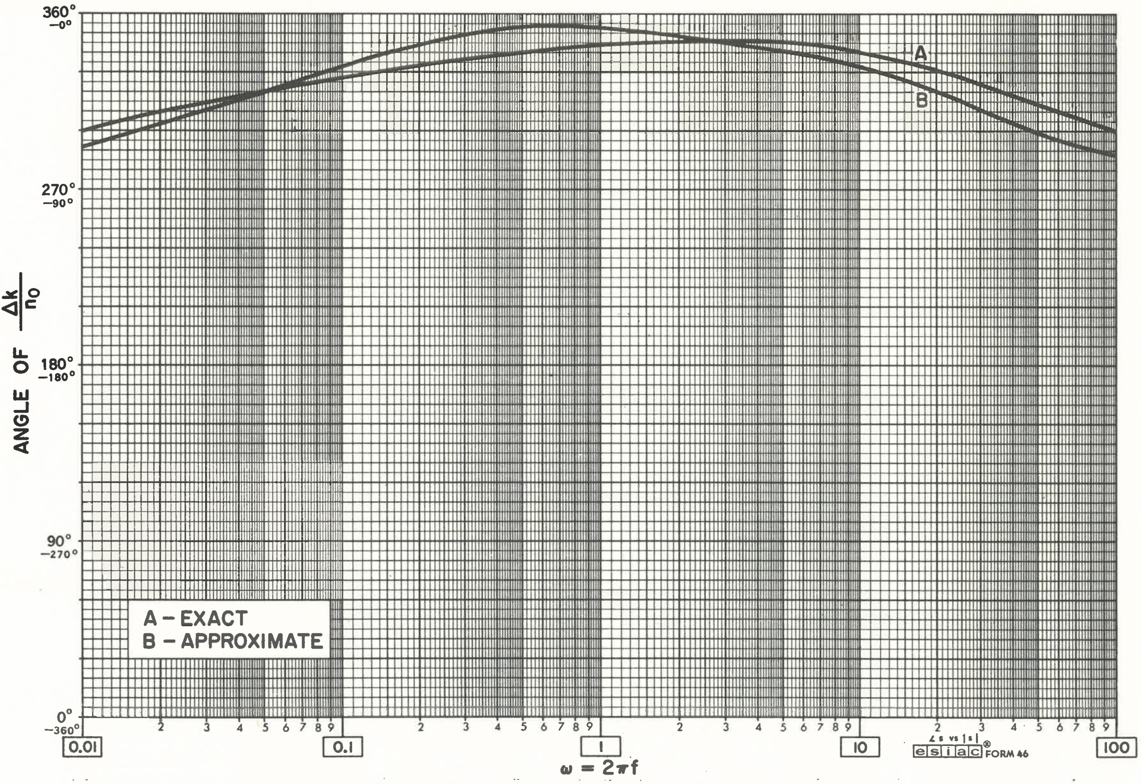

FIGURE 3. FREQUENCY RESPONSE OF ELEMENTARY REACTOR TRANSFER FUNCTION ANGLE

The transfer function:

$$\frac{\Delta n}{\Delta k} = n_o \frac{7(10^3)(s+0.07)}{s(s+35)} = n_o \frac{14\left(1+\frac{s}{0.07}\right)}{s\left(1+\frac{s}{35}\right)} \tag{11}$$

was chosen for this paper and the Bode plots of Equations (9) and (11) are compared in Figures 2 and 3.

The elementary reactor transfer function discussed above ignores two fundamental processes that are also needed to describe reactor operation. These processes produce a poisoning effect and a temperature coefficient effect. The poisoning effect will not be considered in this paper because it is a very low frequency phenomena and is usually most important only during start-up of the reactor after a period of low power operation.

The temperature coefficient effect is primarily due to a decrease in density of the moderator and reflector material as the temperature within the reactor is raised. The change in density of the moderator and reflector usually causes a decrease in reactivity as temperature increases and thus the temperature coefficient is said to be negative. The temperature coefficient is one of the most important reactor parameters and it is usually negative for a graphite-moderated, $U_{235}$ thermal reactor.

The temperature coefficient can be included in the over-all reactor transfer function by adding a feedback loop to the elementary reactor transfer function as shown in Figure 4.

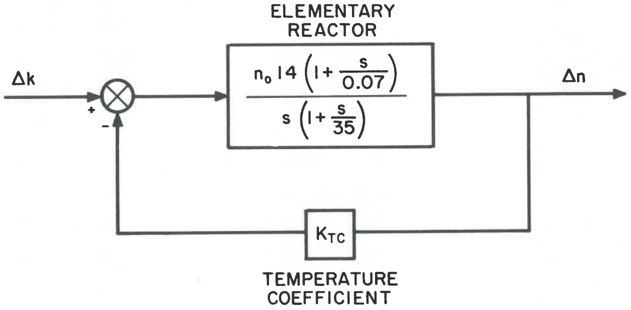
ELEMENTARY REACTOR
Δk
+
-
$n_o 14\left(1+\frac{s}{0.07}\right) / s\left(1+\frac{s}{35}\right)$
Δn
$K_{TC}$
TEMPERATURE COEFFICIENT

FIGURE 4. BLOCK DIAGRAM OF REACTOR WITH TEMPERATURE COEFFICIENT

The reactor transfer function including temperature coefficient can be found by using the feedback equation:

$$\frac{\text{OUTPUT}}{\text{INPUT}} = \frac{G(s)}{1+G(s)H(s)} \tag{12}$$

and is:

$$\frac{\Delta n}{\Delta k} = \frac{14\, n_o\left(1+\frac{s}{0.07}\right)}{s\left(1+\frac{s}{35}\right) + K_{tc}\, 14\, n_o\left(1+\frac{s}{0.07}\right)} \tag{13}$$

Intuitively, it would seem that the negative feedback due to the temperature coefficient would tend to stabilize the reactor. The effect of the reactor negative temperature coefficient on an automatic control system including a reactor is shown in the following example.

## ILLUSTRATIVE EXAMPLE

A block diagram of a nuclear reactor with its associated system components is shown in Figure 5.

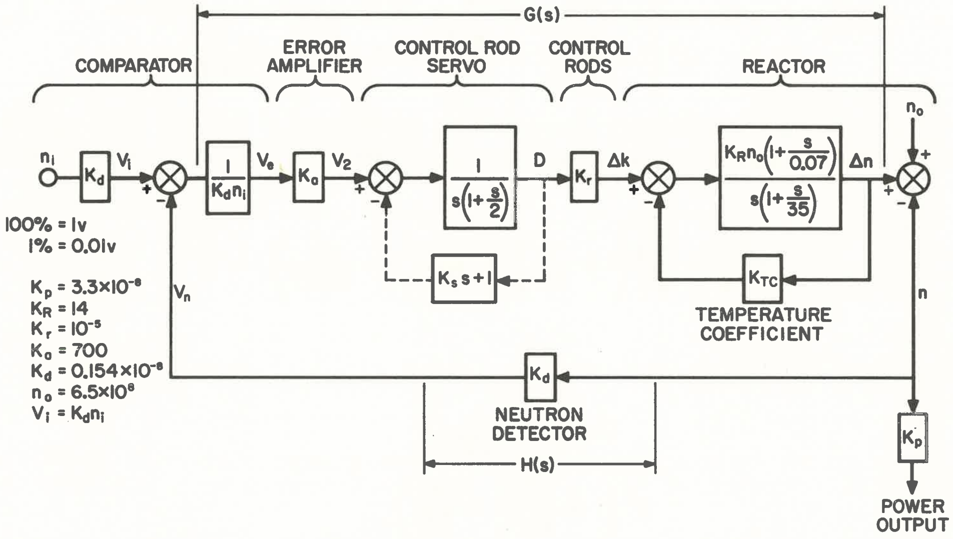

FIGURE 5. NUCLEAR REACTOR AND CONTROL SYSTEM

The output of the system is power which is proportional to the neutron concentration $n$. The neutron concentration consists of a steady-state level $n_o$ and a variation $\Delta n$. The input to the reactor is reactivity $\Delta k$ which is determined by the position of the control rods D. The position of the control rods is determined by the control rod servo which is driven by an error amplifier. The error voltage $V_e$ is the output of the comparator circuit and is described by:

$$V_e = \frac{V_i - V_n}{V_i} \tag{14}$$

This function gives an error voltage that is inversely proportional to steady-state power level $n_o$, since $V_i = k_d n_i$, and $n_i$ is approximately equal to $n_o$. This inverse relationship is deliberately used to cancel out the effect of the $n_o$ in the reactor transfer function. Unfortunately, the amount of cancellation depends upon the temperature coefficient $K_{tc}$ as shown later.

We have said previously that the reactor temperature coefficient is likely to be negative for a graphite-moderated thermal reactor. To begin the analysis of the control system, let us assume that the temperature coefficient may vary from zero to some negative value and that the system will be least stable for the zero value. We can find the stability characteristics of the system for this worst condition and then find the change in the system due to the temperature coefficient becoming negative.

The complete system transfer function with $K_{tc} = 0$ is derived in the appendix and is:

$$\frac{n}{n_i} = \frac{1 + G(s)}{1 + G(s)H(s)} \tag{15}$$

Substituting the values of $G(s)$ and $H(s)$ and simplifying,

$$\frac{n}{n_i} = \frac{0.154\left(10^{-8}\right)\left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{35}\right) + 0.1\left(1 + \frac{s}{0.07}\right)}{s^2\left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{35}\right) + 0.1\left(1 + \frac{s}{0.07}\right)} \tag{16}$$

the poles of $\frac{n}{n_i}$ determine whether or not the system is stable and these can be found by setting the denominator equal to zero and finding the roots of the resulting equation:

$$s^2\left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{35}\right) + 0.1\left(1 + \frac{s}{0.07}\right) = 0 \tag{17}$$

or,

$$\frac{0.1\left(1 + \frac{s}{0.07}\right)}{s^2\left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{35}\right)} = -1 \tag{18}$$

using the root-locus technique and an ESIAC potential-plane analog, we obtain the root-locus plot shown in Figure 6.

Figure 6 shows that the poles of $\frac{n}{n_i}$ are located in the stable region of the log s plane and their values are:

$P_1 = -35$
$P_2 = -0.076$
$P_3 = 1.5\angle 120$
$P_4 = 1.5\angle -120$

The damping factor for the complex poles is found from the damping factor scale and is 0.5.

The zeros of $\frac{n}{n_i}$ can be found by applying the above technique to the numerator of Equation (16). The equation to be solved is:

$$\frac{6.5\left(10^{7}\right)\left(1 + \frac{s}{0.07}\right)}{s^2\left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{35}\right)} = -1 \tag{19}$$

This is the same as Equation (18) except for the multiplying constant so the root-locus plot shown in Figure 6 can also be used to find the zeros.

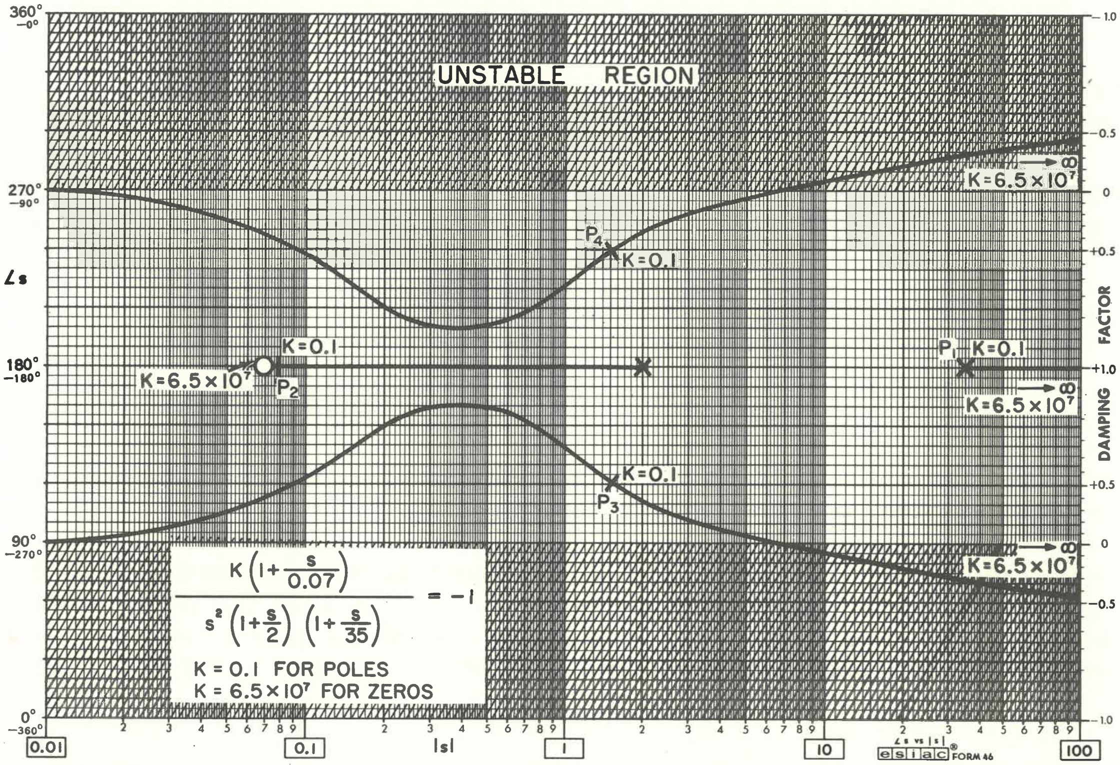

FIGURE 6. LOG s PLANE ROOT LOCUS PLOT OF REACTOR CONTROL SYSTEM WITH ZERO TEMPERATURE COEFFICIENT

As far as stability is concerned, there is no restriction on the location of the zeros and the values are seen to be:

$$Z_1 = -0.07, \quad Z_2, \quad Z_3, \quad Z_4, \quad \text{AT } \infty$$

Using these pole and zero values, the complete transfer function with $K_{tc} = 0$ is:

$$\frac{n}{n_i} = \frac{\cancel{\left(1+\frac{s}{0.07}\right)}}{\left(1-\frac{s}{1.5\angle 120}\right)\left(1-\frac{s}{1.5\angle -120}\right)\left(1+\frac{s}{35}\right)\cancel{\left(1+\frac{s}{0.076}\right)}} \tag{20}$$

Notice that $n_0$ does not appear in the transfer function which means that the system behavior does not depend on the steady state power level. However, this may not be true when the temperature coefficient is other than zero.

Let us now consider what happens to the control system if the temperature coefficient $K_{tc}$ is not zero. The complete reactor transfer function is given in Equation (13) and when it is incorporated in the control system, the overall closed loop transfer function is:

$$\frac{n}{n_i} = \frac{K_d s^2\left(1+\frac{s}{2}\right)\left(1+\frac{s}{35}\right) + n_i K_{tc} K_R K_d \left(1+\frac{s}{2}\right)\left(1+\frac{s}{0.07}\right) + 0.1\left(1+\frac{s}{0.07}\right)}{s^2\left(1+\frac{s}{2}\right)\left(1+\frac{s}{35}\right) + n_i K_{tc} K_R s \left(1+\frac{s}{2}\right)\left(1+\frac{s}{0.07}\right) + 0.1\left(1+\frac{s}{0.07}\right)} \qquad (21)$$

The intermediate steps are shown in the appendix. Notice that when $K_{tc} = 0$, Equation (12) reduces to Equation (16).

The poles of this system are found by setting the denominator of Equation (21) equal to zero and solving for the values of s that satisfy the resulting equation.

$$-1 = \frac{n_i K_{tc} 140 s \left(1+\frac{s}{2}\right)\cancel{\left(1+\frac{s}{0.07}\right)}}{0.1\left(1-\frac{s}{1.5\angle 120}\right)\left(1-\frac{s}{1.5\angle -120}\right)\left(1+\frac{s}{35}\right)\cancel{\left(1+\frac{s}{0.076}\right)}} \qquad (22)$$

Here again, the easiest way to solve the above equation is to use the root-locus technique and an ESIAC potential plane analog.

Figure 7 shows the root locus plot of Equation (22) as obtained with an ESIAC with the product $n_i K_{tc}$ as the varying parameter. Notice that the poles do not enter the unstable region as $K_{tc}$ is varied. However, one of the poles does approach the origin and for any reasonable value of $K_{tc}$ can be considered to be at the origin.

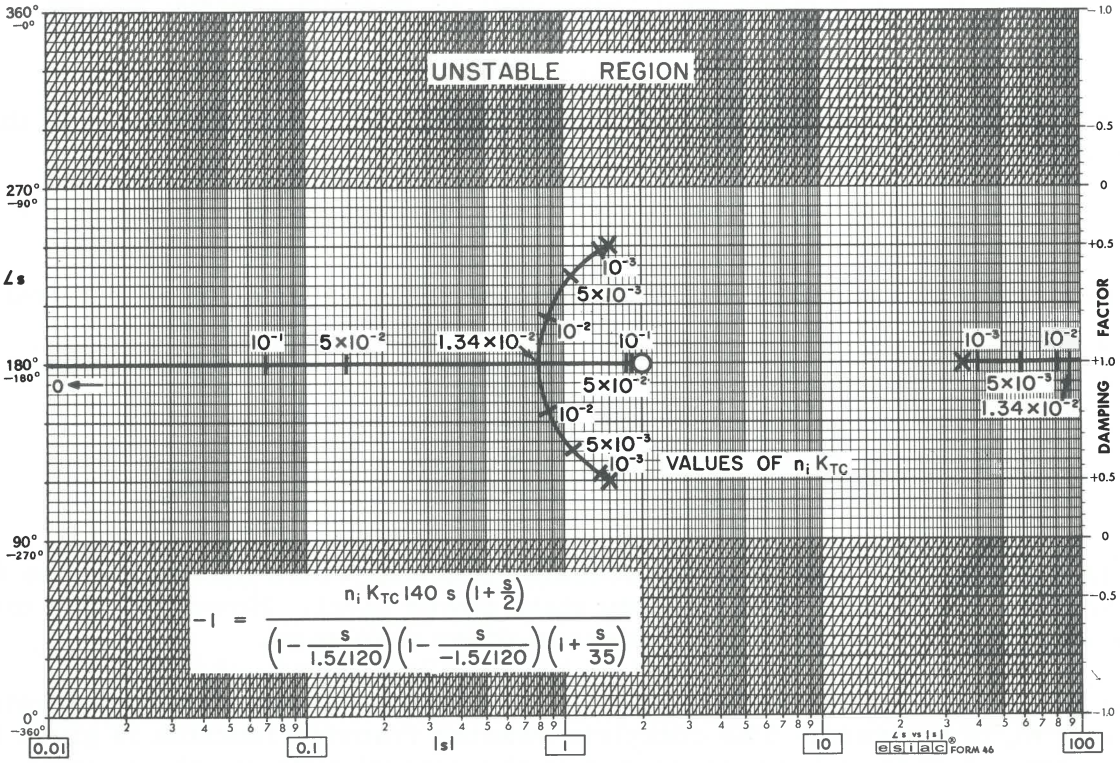

FIGURE 7. LOG s PLANE ROOT LOCUS PLOT OF REACTOR CONTROL SYSTEM WITH NEGATIVE TEMPERATURE COEFFICIENT

The reference $n_i$ should be approximately equal to $n_0$ which is on the order of $10^8$ neutrons per cubic centimeter. This means that when $K_{tc}$ is $10^{-8}$ or greater, the poles of the system will be at:

$$s_1 = 0$$
$$s_2 = -2$$
$$s_3 \approx -00$$

which gives:

$$\frac{n}{n_i} = \frac{\text{ZEROS}}{s\left(1+\frac{s}{2}\right)} \qquad (23)$$

Although not shown here, the zeros are found in the same manner by setting the numerator of Equation (21) equal to zero and solving for the roots. The result gives:

$$\frac{n}{n_i} = \frac{\left(1-\frac{s}{35\angle 110}\right)\left(1-\frac{s}{35\angle -110}\right)}{s\left(1+\frac{s}{2}\right)} \qquad (24)$$

The fact that one of the poles of $\frac{n}{n_i}$ is at the origin is usually undesirable so the system should be modified. Notice that the poles of the system are identical to the poles of the control rod servo. This suggests that the system might be improved by changing the characteristics of the servo, as for example, by adding tachometer feedback. This modification is shown in Figure 5 by the dotted feedback loop around the control rod servo. The question immediately arises as to what value of tachometer constant $K_s$ should be used?

The control rod servo transfer function $\frac{D}{V_2}$ becomes:

$$\frac{D}{V_2} = \frac{1}{s\,K_s + 1 + s\left(1+\frac{s}{2}\right)} \qquad (25)$$

which can be solved with a root locus plot. However, this is only a quadratic so using the value of $K_s$ that gives critical damping, the control rod servo transfer function becomes: $(K_s = 1.83)$

$$\frac{D}{V_2} = \frac{1}{\left(1+\frac{s}{1.4}\right)\left(1+\frac{s}{1.4}\right)} \qquad (26)$$

This new transfer function can now be used in place of the original control rod servo transfer function in the deviations of Equations (16) and (21). The new pole locations of the complete system are found as shown above for $K_{tc} = 0$ and $K_{tc}$ negative. The results are shown in Figures 8 and 9. Notice in Figure 9 that when $K_{tc}$ differs from zero the poles of the system do not enter the unstable region or approach the origin.

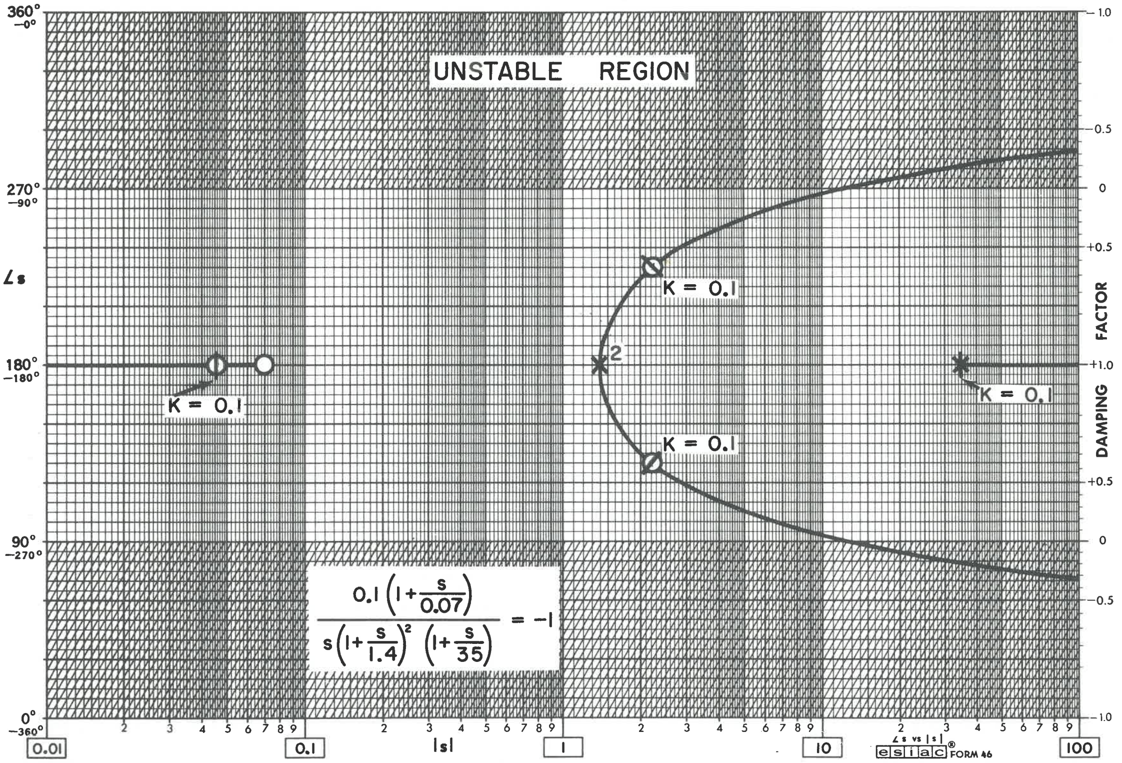

FIGURE 8. LOG s PLANE ROOT LOCUS PLOT OF REACTOR CONTROL SYSTEM WITH ZERO TEMPERATURE COEFFICIENT AND MODIFIED CONTROL ROD SERVO

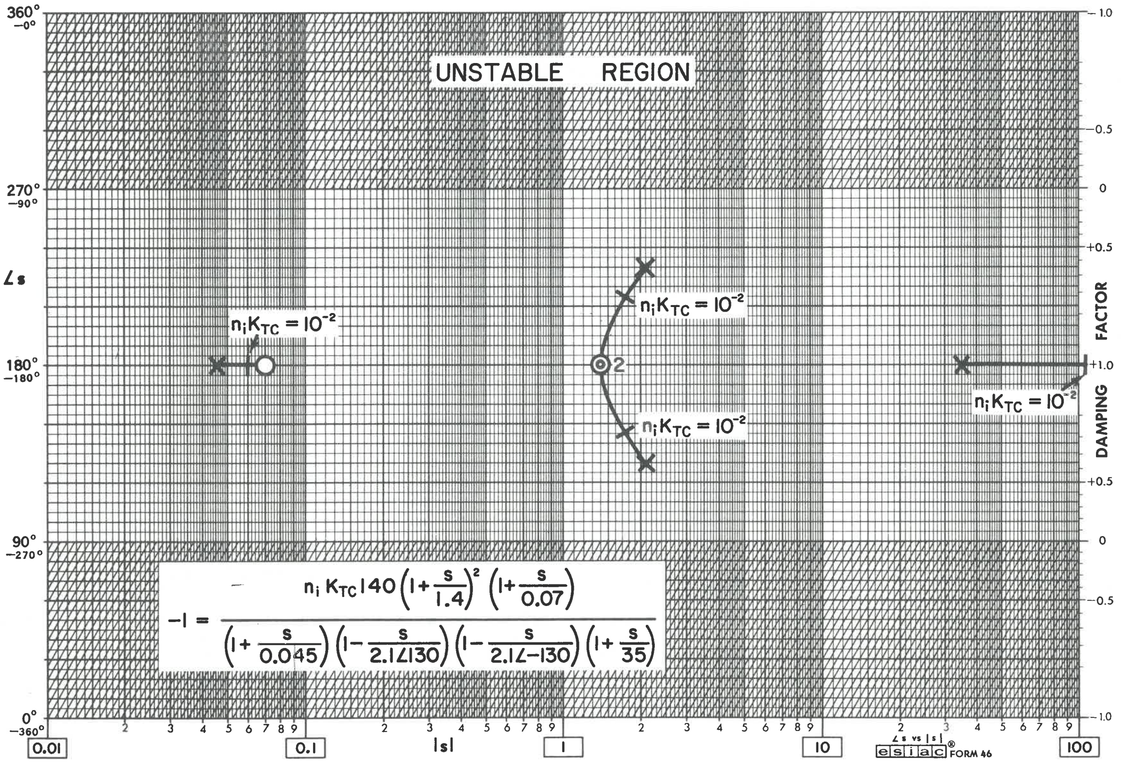

FIGURE 9. LOG s PLANE ROOT LOCUS PLOT OF REACTOR CONTROL SYSTEM WITH NEGATIVE TEMPERATURE COEFFICIENT AND MODIFIED CONTROL ROD SERVO

## SUMMARY

A potential plane analog of the log s plane can be used to great advantage for investigating problems involving reactor transfer functions and control systems. These uses include four fundamental operations:

(1) Root Locus Plots
(2) Bode Plots
(3) Residue Evaluation
(4) Polynomial Factorization

Although not illustrated in the above example, these operations can also be applied to problems involving:

(a) Dead Time
(b) Sampled-Data System
(c) Linear Systems
(d) Non-Linear Systems

and others.

The above example has illustrated the use of the potential plane analog for: root locus plots, (Figures 6, 7, 8, 9); Bode plots, (Figures 2, 3) and factoring polynomials, (Equations 18, 19, 22).

## REFERENCES

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# APPENDIX

## DERIVATION OF REACTOR TRANSFER FUNCTION
(Reference 5)

Since $\sum_{i=1}^{6} \beta_i = \beta$ Equations (5) and (6) are combined to get:

$$\frac{dn}{dt} = \frac{n\Delta k}{T} - \frac{n\beta}{T} + \sum_{i=1}^{6}\left[\frac{n\beta_i}{T} - \frac{dC_i}{dt}\right]$$

$$\frac{dn}{dt} = \frac{n\Delta k}{T} - \sum_{i=1}^{6}\frac{dC_i}{dt} \qquad \text{(A1)}$$

Now let $n$ and $C_i$ be separated into two parts; the steady-state values $n_o$ and $C_{io}$, and the small variations $\Delta n$ and $\Delta C_i$.

Then $$\frac{dn}{dt} = \frac{d\Delta k}{dt} = \frac{n_o\Delta k}{T} + \frac{\Delta n\Delta k}{T} - \sum_{i=1}^{6}\frac{d\Delta C_i}{dt} \qquad \text{(A2)}$$

Since $$\frac{dn_o}{dt} = 0, \qquad \frac{dC_{io}}{dt} = 0$$

and $\frac{\Delta n\Delta k}{T}$ is small compared to $\frac{n_o\Delta k}{T}$.

also $$\frac{dC_i}{dt} = \frac{d\Delta C_i}{dt} = \frac{n_o\beta_i}{T} + \frac{\Delta n\beta_i}{T} - \lambda_i C_{io} - \lambda_i\Delta C_i$$

which is $$\frac{d\Delta C_i}{dt} = \frac{n_o\beta_i}{T} - \lambda_i\Delta C_i \qquad \text{(A3)}$$

since $$dC_{io} = 0 = \frac{n_o\beta_i}{T} - \lambda_i C_{io}$$

Now take the Laplace Transform of (A2) and (A3) to get:

$$s\Delta n(s) = \frac{n_o\Delta k(s)}{T} - s\sum_{i=1}^{6}\Delta C_i(s) \qquad \text{(A4)}$$

$$s\Delta C_i(s) = \frac{\beta_i\Delta n(s)}{T} - \lambda_i\Delta C_i(s) \qquad \text{(A5)}$$

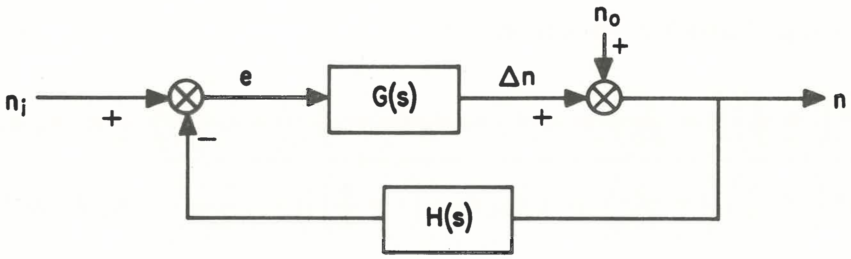
where initial conditions are equal to zero. Rearranging (A5) gives:

$$\Delta C_i(s) = \frac{\beta_i\Delta n(s)}{T(s+\lambda_i)} - \lambda_i\Delta C_i(s)$$

which can be substituted into (A4) and rearranged to get Equation (7).

$$\frac{\Delta n(s)}{\Delta k(s)} = \frac{n_o}{T} \frac{1}{s\left[1 + \sum_{i=1}^{6} \frac{\beta_i}{T(s+\lambda_i)}\right]} \tag{7}$$

## DERIVATION OF CLOSED-LOOP TRANSFER FUNCTION



$$n_i = n_{io} + \Delta n_i$$
$$n = n_o + \Delta n = n_o + eG(s)$$
$$e = n_i - nH(s)$$
$$n = n_o + G(s)\left[n_i - nH(s)\right]$$
$$= n_o + n_i G(s) - n\,G(s)\,H(s)$$

$$n\left(1+G(s)H(s)\right) = n_o + n_i G(s)$$

$$\frac{n}{n_i}\left(1+G(s)\,H(s)\right) = \frac{n_o}{n_i} + G(s) \cong 1 + G(s)$$

because $n_i \cong n$ and $n_{io} \gg \Delta n_i$.

$$\frac{n}{n_i} = \frac{1+G(s)}{1+G(s)H(s)} \tag{15}$$

Referring to Figure 5:

$$H(s) = K_d$$

$$G(s) = \left(\frac{\Delta n}{\Delta k}\right) \frac{K_r\,K_a}{K_d\,n_i} \frac{1}{s\left(1+\frac{s}{2}\right)}$$

For $K_{tc} \neq 0$:

$$\frac{\Delta n}{\Delta k} = \frac{n_o K_R \left(1 + \frac{s}{0.07}\right)}{s\left(1 + \frac{s}{35}\right) + n_o K_R K_{tc} \left(1 + \frac{s}{0.07}\right)}$$

Then:

$$G(s) = \frac{K_r K_a K_R \left(1 + \frac{s}{0.07}\right)}{\frac{n_i}{n_o} K_d s^2 \left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{35}\right) + n_i K_{tc} K_R K_d s \left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{0.07}\right)}$$

and the closed-loop transfer function is:

$$\frac{n}{n_i} = \frac{K_d s^2 \left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{35}\right) + n_i K_{tc} K_R K_d s \left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{0.07}\right) + K_r K_a K_R \left(1 + \frac{s}{0.07}\right)}{s^2 \left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{35}\right) + n_i K_{tc} K_R s \left(1 + \frac{s}{2}\right)\left(1 + \frac{s}{0.07}\right) + K_r K_a K_R \left(1 + \frac{s}{0.07}\right)}$$

which is Equation (21) when the values of the constants are included.

If $K_{tc} = 0$, the center terms in the numerator and denominator of Equation (21) drop out and the result is Equation (16).